C3.Q102.NOTES CH 14A

LESSON 1 (14.1)

Definition of: **a function *f* of two variables**

Example 1: Let $f(x,y) = \dfrac{xy-5}{2\sqrt{y-x^2}}$

(a) Find and sketch the domain *D* of *f*.
(b) Evaluate $f(2,5)$, $f(1,2)$, $f(-1,2)$

Discuss a function of three variables (ex: *D* solid and *f* temp)

Example 2: Let *f* be the function with domain *D* given by $f(x,y) = 9 - x^2 - y^2$ and $D = \{(x,y): x^2 + y^2 \le 9\}$

(a) Sketch the graph of *f* and show the **traces** on the planes z = 0, z = 5, z = 8.
(b) Sketch some **level curves** of *f*.

Discuss the difference between traces and level curves

(c) Sketch the *xz* (*y* = 0) and *yz* (*x* = 0) trace

Example 3: If $f(x,y) = y^2 - x^2$ sketch some level curves of *f*.
(Consider the Cartesian quadratic relationship when $k > 0$, $k < 0$, $k = 0$)

Applications (Level Curves):
Topographic, or contour, maps
Water Depth
High-Temperature: Isothermal curves, or isotherms
Barometric Pressure: Isobars
Cost and Profit Functions: Isoquants (isocost / isoprofit)
Probability Density Functions

Example 4: If $f(x,y,z) = z - \sqrt{x^2 + y^2}$, sketch some level surfaces of *f*.
Example 5: If $f(x,y,z) = x^2 - y^2 + z^2$, discuss level curves when $k > 0$, $k = 0$, $k < 0$

Application:
Temperature: isothermal surface
Electric Potential: equipotential surface

Discuss Computer Graphics: TI84, GRAPHMATICA, DERIVE, MATLAB, MAPLE

LESSON 2 (14.3, 14.5)

Definition: Let f be a function of two variables.

The **first partial derivative of f with respect to x** is the function

$f_x(x, y) = \lim_{h\to 0} \dfrac{f(x+h, y) - f(x, y)}{h}$ provided it exists.

The **first partial derivative of f with respect to y** is the function

$f_y(x, y) = \lim_{h\to 0} \dfrac{f(x, y+h) - f(x, y)}{h}$ provided it exists.

Notation: If $z = f(x, y)$, then $f_x = \dfrac{\partial f}{\partial x} = \dfrac{\partial z}{\partial x}$ and $f_y = \dfrac{\partial f}{\partial y} = \dfrac{\partial z}{\partial y}$

Example1: $f(x, y) = x^3y^2 - 2x^2y + 3x$

Example2: $z = xy^2e^{xy}$

Understanding Partial derivatives: [Include Drew's – Power Point]

Example3: $w = x^2y^3 \sin z + e^{xz}$

Second Partial Derivatives:

$$\frac{\partial}{\partial x} f_x = (f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial}{\partial y} f_x = (f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial}{\partial x} f_y = (f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial}{\partial y} f_y = (f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Clairaut's Theorem: Let f be a function of tow variables x and y. If f, f_x, f_y, f_{xy}, f_{yx} are continuous on an open interval region R, then $f_{xy} = f_{yx}$ throughout R.

Example 4: $f(x, y) = x^3y^2 - 2x^2y + 3x$

Example 5: $w = r^3 + s^2$ with $r = pq^2$ and $s = p^2 \sin q$

Example6: $w = r^2 + sv + t^3$, with $r = x^2 + y^2 + z^2$, $s = xyz$, $v = xe^y$, $t = yz^2$

Example7: $w = x^2 + yz$, with $x = 3t^2 + 1$, $y = 2t - 4$, $z = t^3$ Find $\frac{dw}{dt}$.

Chain Rule for Partial Derivatives:

Diagram for Chain Rule:

Review: Find $f'(x) = \frac{dy}{dx}$ if $y = f(x)$ is determined implicitly by $y^4 + 3y - 4x^3 - 5x = 0$.

Discover: Consider any equation $F(x, y) = 0$. Let $z = F(u, y)$ where $u = x$ and $y = f(x)$.
Find $f'(x) = \frac{dy}{dx}$.

> Theorem: If an equation $F(x, y) = 0$ determines, implicitly, a differentiable function f of one variable x such that $y = f(x)$, then
> $$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}.$$

Example 8: Use partial derivatives to find $f'(x) = \frac{dy}{dx}$ if $y = f(x)$ is determined implicitly by $y^4 + 3y - 4x^3 - 5x = 0$.

LESSON 3 (14.6)

Example 1: Let $f(x,y)=x^3y^2$

(a) Find the directional derivative of *f* at the point P(-1,2) in the direction of the vector $\mathbf{v}=\langle 4,-3\rangle$.

(b) Discuss the significance of part (a) if *f*(*x*,*y*) is the temperature at (*x*,*y*).

Example 2: Let $f(x,y)=x^2-4xy$

(a) Find the gradient of *f* at the point P(1,2) , and sketch this vector.

(b) Use the gradient vector to find the directional derivative of *f* at P(1,2) in the direction from P(1,2) to Q(2,5).

Example 3: Let $f(x,y)=2+x^2+\frac{1}{4}y^2$

Find the direction in which *f*(*x*,*y*) increases most rapidly at the point P(1,2), and find the maximum rate of change of *f* at P.

Example 4: Suppose an *xyz*-coordinate system is located in space such that temperature T at the point (*x*,*y*,*z*) is given by $T=100/\left(x^2+y^2+z^2\right)$.

(a) Find the rate of change of T with respect to distance at the point P(1,3,-2) in the direction of the vector <1,-1,1>.

(b) In what direction from P does T increase most rapidly? What is the maximum rate of change of T at P?

Example 5: Write an equation of a plane tangent to surface $x^2+3y^2-z^2=0$ at the point P(1,1,2)